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ON THE ASYMPTOTIC BEHAVIOR OF JACOBI POLYNOMIALS WITH VARYING PARAMETERS

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ABSTRACT. We study the large n behavior of Jacobi polynomials with varying parameters $P_n^{(an+\alpha,\beta)}(1-2\lambda^2)$ for $a>-1$ and $\lambda\in(0,1)$. This appears to be a well-studied topic in the literature but some of the published results are unnecessarily complicated or incorrect. The purpose of this paper is to provide a simple and clear discussion and to highlight some flaws in the existing literature. Our approach is based on a new representation for $P_n^{(an+\alpha,\beta)}(1-2\lambda^2)$ in terms of two integrals. The integrals' asymptotic behavior is studied using standard tools of asymptotic analysis: one is a Laplace integral and the other is treated via the method of stationary phase. In summary we prove that if $a\in(\frac{2\lambda}{1-\lambda},\infty)$ then $\lambda^{an}P_n^{(an+\alpha,\beta)}(1-2\lambda^2)$ shows exponential decay and we derive exponential upper bounds in this region. If $a\in(\frac{-2\lambda}{1+\lambda},\frac{2\lambda}{1-\lambda})$ then the decay of $\lambda^{an}P_n^{(an+\alpha,\beta)}(1-2\lambda^2)$ is $\mathcal{O}(n^{-1/2})$ and if $a\in\{\frac{-2\lambda}{1+\lambda},\frac{2\lambda}{1-\lambda}\}$ then $\lambda^{an}P_n^{(an+\alpha,\beta)}(1-2\lambda^2)$ decays as $\mathcal{O}(n^{-1/3})$. Lastly we find that if $a\in(-1,\frac{-2\lambda}{1+\lambda})$ then $\lambda^{an}P_n^{(an+\alpha,\beta)}(1-2\lambda^2)$ decays exponentially iff $an+\alpha$ is an integer and increases exponentially iff it is not.

1. Introduction and Summary

Jacobi polynomials $P_n^{(\alpha,\beta)}$ constitute a class of orthogonal degree-n polynomials, which depend on two parameters $\alpha,\beta\in\mathbb{R}$. An explicit representation is [GZ, (4.3.2) p. 68]

$$P_n^{(\alpha,\beta)}(x)=\sum_{\mu=0}^{n}\binom{n+\alpha}{n-\mu}\binom{n+\beta}{\mu}\left(\frac{x-1}{2}\right)^{\mu}\left(\frac{x+1}{2}\right)^{n-\mu}.$$

Historically Jacobi polynomials have experienced exhaustive research, see for example [WW, GZ] for an overview of existing results. In this paper we are concerned with with the behavior of Jacobi polynomials for large n, which is a classical topic in asymptotic analysis. To gain some background we state the probably most famous result: Darboux's asymptotic formula, which we cite in the formulation of Szegö [GZ, Theorem 8.21.8, p. 196].

Proposition 1. *Let $\alpha,\beta\in\mathbb{R}$ and let $\theta\in[\epsilon,\pi-\epsilon]$ then the Jacobi polynomials satisfy the following asymptotic expansion*

$$P_n^{(\alpha,\beta)}(\cos(\theta))=\frac{1}{\sqrt{n\pi}}k(\theta)\cos(N\theta+\gamma)+\mathcal{O}\left(\frac{1}{n^{3/2}}\right),$$

where

$$k(\theta)=\frac{1}{\sqrt{\pi}}\left(\sin\frac{\theta}{2}\right)^{-\alpha-\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}},$$

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$$N = n + (\alpha+\beta+1)/2,$$
$$\gamma = -(\alpha+\frac{1}{2})\pi/2, 0 < \theta < \pi.$$

The main assertion of Darboux's formula is that for fixed α, β Jacobi polynomials decay for large n in first order like $\mathcal{O}(n^{-1/2})$. The asymptotic growth of Jacobi polynomials with varying parameters,

$$\lambda^{an} P_n^{(\alpha+an,\beta)}(1-2\lambda^2)$$

where $-1 < \alpha, \beta, a$, is studied in multiple articles, see e.g. [BG, CI, FFN, GS, SI, SV]. The behavior is non-trivial in that within the a-range $(-1,\infty)$ different types of asymptotic bounds occur. For instance, it is the main result of [BG, GS, CI, SI] that for $a \in (-\frac{2\lambda}{1+\lambda}, \frac{2\lambda}{1-\lambda})$ in first order a $\mathcal{O}(n^{-1/2})$ asymptotic behavior can be observed but the explicit formulas are inconsistent. The main point of our article is to tidy up the inconsistent and partially incorrect findings in the literature and provide a simple access to the topic. We begin by stating our main result and relating it to the existing literature.

Theorem 2. *Let $\alpha,\ \beta > -1$, $a > -1$ and $\lambda \in (0,1)$. We have the following asymptotic expansion as $n \to \infty$:*

(1) *If $a \in (-\frac{2\lambda}{1+\lambda}, \frac{2\lambda}{1-\lambda})$ then*

$$P_n^{(\alpha+an,\,\beta)}(1-2\lambda^2) = \sqrt{\frac{2}{n\pi}} \frac{\lambda^{-\alpha-an}((1-\lambda^2)(a+1))^{-\frac{\beta}{2}}}{((1-\lambda^2)((a+2)\lambda+a)((a+2)\lambda-a))^{\frac{1}{4}}}$$
$$\cdot \cos\Big((n+1)h(\varphi_+) + (\alpha-a)\varphi_+ + (\beta-1)\psi + \frac{\pi}{4}\Big)\left(1+\mathcal{O}(n^{-1})\right).$$

The phases $\varphi_+, h(\varphi_+), \psi \in [0,2\pi]$ depend on a, λ and are given explicitly in Proposition 4 below.

(2) *If $a = \frac{2\lambda}{1-\lambda}$ then*

$$P_n^{(an+\alpha,\,\beta)}(1-2\lambda^2) = \frac{\lambda^{-an-\alpha}(1+\lambda)^{-\beta}}{3^{2/3}\Gamma(2/3)n^{1/3}\lambda^{1/3}(1+\lambda)^{1/3}}\left(1+\mathcal{O}(n^{-1/3})\right).$$

If $a = -\frac{2\lambda}{1+\lambda}$ then

$$\begin{aligned} P_n^{(an+\alpha,\,\beta)}(1-2\lambda^2) &= \frac{\Gamma(1/3)\lambda^{-\alpha-an}(1-\lambda)^{-\beta}}{3^{2/3}\pi n^{1/3}\lambda^{1/3}(1-\lambda)^{1/3}} \\ &\cdot\frac{\sqrt{3}}{2}\Big(\cos\left((an+\alpha)\pi\right) - \sqrt{3}\sin\left((an+\alpha)\pi\right)\Big)\left(1+\mathcal{O}(n^{-1/3})\right). \end{aligned}$$

(3) *If $a \in (-1, \frac{-2\lambda}{1+\lambda}) \cup (\frac{2\lambda}{1-\lambda}, \infty)$ and $an+\alpha$ is integer then the quantity*

$$\lambda^{an+\alpha}(1-\lambda^2)^{\beta} P_n^{(an+\alpha,\,\beta)}(1-2\lambda^2)$$

decays exponentially with n, see Proposition 7 for details.

(4) *If $a \in (-1, -\frac{2\lambda}{1+\lambda})$ and $an+\alpha$ is not integer then the quantity*

$$\lambda^{an+\alpha}(1-\lambda^2)^{\beta} P_n^{(an+\alpha,\,\beta)}(1-2\lambda^2)$$

increases exponentially with n, see Proposition 7 for details. If $a \in (\frac{2\lambda}{1-\lambda}, \infty)$ and $an+\alpha$ is not an integer then the above quantity decays exponentially, see Proposition 7 for details.

The same topic was studied in earlier publications. Chen, Ismail and Izen [CI, SI] derive asymptotic bounds as in Theorem 2 but their findings are inaccurate in the whole parameter range of a. [GS, FFN] derive (among other things) the asymptotic behavior in the range $a \in (-\frac{2\lambda}{1+\lambda}, \frac{2\lambda}{1-\lambda}) \cup (\frac{2\lambda}{1-\lambda}, \infty)$ and in this range we recover their findings using our new method. In the range $a \in (-1, \frac{-2\lambda}{1+\lambda})$ [SI] claim exponential decay for $\lambda^{an} P_n^{(\alpha+an,\beta)}(1-2\lambda^2)$ but we find that this happens iff $an+\alpha$ is integer. In other words exponential decay occurs along a subsequence of values n, where $an+\alpha$ is integer. If $an+\alpha$ is not integer we prove that $\lambda^{an} P_n^{(\alpha+an,\beta)}(1-2\lambda^2)$ is increasing exponentially. Finally we derive the asymptotic behavior at the saddle points $a \in \{\frac{-2\lambda}{1+\lambda}, \frac{2\lambda}{1-\lambda}\}$, where the asymptotics were not studied before.

Various methods are employed in the cited publications to access the problem. Gawronksi-Shawyer [GS] as well as Saff-Varga [SV] rely on the *method of steepest descent* [BO, p. 147] [FO0, p. 136], while Chen-Ismail [CI] and Izen [SI] make use of *Darboux's asymptotic method* and generating functions [FO0, GZ]. In this article we derive a simple integral representation for the Jacobi polynomials with varying parameters, see Lemma 3. The lemma provides a representation of $\lambda^{an} P_n^{(\alpha+an,\beta)}(1-2\lambda^2)$ as a sum of two integrals, which can be treated using standard tools from asymptotic analysis:

(1) The first integral is a so-called *generalized Fourier integral* [AE], [BO, Chapter 6.5], which is of the form

$$\int_a^b g(t) e^{inh(t)} \mathrm{d}t,$$

with continuous real functions g, h. To determine the leading order behavior we employ the method of stationary phase, which is commonly used to study asymptotic properties of oscillatory integrals, see Section 3.1. The method is similar to *Laplace's method* (see below) in that the leading order contribution comes from a small neighborhood around the stationary points of h. Our analysis relies on the well-known fact [AE, Theorem 4] [BO, Chapter 6.5] that if ξ is a stationary point of h such that $g(\xi) \neq 0$ then
  (a) the integral goes to 0 as $n^{-1/2}$ if $h''(\xi) \neq 0$,
  (b) the integral goes to 0 as $n^{-1/3}$ if $h''(c) = 0$ but $h^{(3)}(c) \neq 0$.

(2) The second integral contributes iff $\alpha + an$ is not an integer and is a so-called Laplace type integral [FO0, Chapter 3], which is of the form

$$\int_a^b g(t) e^{nh(t)} \mathrm{d}t,$$

with continuous, real functions g, h. We use Laplace's method to determine the asymptotic behavior, see Section 3.2. The idea is that if the real continuous function h has a maximum at a point $\xi \in [a, b]$ and if $g(\xi) \neq 0$ then as n grows only values in an immediate neighborhood of ξ should contribute to the integral, see [BO, FO0].

2. Integral representation for Jacobi Polynomials

The main point in our approach is to find a good integral representation for Jacobi Polynomials. The emerging integrals are treated using standard methods from asymptotic analysis [BO, FO0, AE]. Although simple, we regard the below lemma as the key technical innovation of our work.

Lemma 3. *[Integral representation for Jacobi polynomials] Let n be an integer, $a > -1$, α, $\beta > -1$, $\lambda \in (0,\, 1)$. For any $x \in (\lambda,\, 1/\lambda)$ we have the following integral representation for the Jacobi polynomials with varying parameters*

$$\begin{aligned}
&\lambda^{an+\alpha}(1-\lambda^2)^{\beta} P_n^{(an+\alpha,\,\beta)}(1-2\lambda^2) \\
&= \frac{1}{\pi}\Re\left\{ \int_0^{\pi} z^{\alpha+1}\frac{(1-\lambda z)^{\beta}}{z-\lambda}\left(z^{a+1}\frac{1-\lambda z}{z-\lambda}\right)^n \bigg|_{z=xe^{i\varphi}} \mathrm{d}\varphi \right\} \\
&- \frac{\sin\left(\pi(\alpha+an)\right)}{\pi}\int_0^x \frac{(1+\lambda t)^{\beta} t^{\alpha}}{t+\lambda}\left(t^{(a+1)}\frac{1+\lambda t}{t+\lambda}\right)^n \mathrm{d}t,
\end{aligned}$$

where $\Re(\bullet)$ refers to the real part of a complex number.

This representation is particularly suited for asymptotic analysis. The first integral is a generalized Fourier integral and its large n behavior is derived using the method of stationary phase. The second one is a Laplace integral and can be studied using Laplace's asymptotic results. Curiously the second integral contributes iff $an+\alpha$ is not an integer. As a consequence it is necessary to consider two different types of subsequences of values n in the study of the asymptotic behavior of $\lambda^{an+\alpha}(1-\lambda^2)^{\beta} P_n^{(an+\alpha,\,\beta)}(1-2\lambda^2)$. For a subsequence of n, where $an+\alpha$ is integer the second integral does not contribute but if $an+\alpha$ is not an integer the second integral contributes and changes the asymptotic behavior of $\lambda^{an+\alpha}(1-\lambda^2)^{\beta} P_n^{(an+\alpha,\,\beta)}(1-2\lambda^2)$ "discontinuously". A similar phenomenon was observed in [KM] for $a \in (-1,0)$ and $\beta = bn$ with $b \in (-1,0)$ and $a+b < -1$. In particular it suggests that the limiting behavior of zeros of the corresponding Jacobi polynomials [MMO] will be very sensitive to the proximity of an, bn, $an+bn$ to integers [KM].

Proof. We investigate the complex function

$$f : z \mapsto z^{(a+1)n+\alpha}\frac{(1-\lambda z)^{n+\beta}}{(z-\lambda)^{n+1}},$$

where powers are taken with respect to the principal branch. We integrate f along a closed contour γ in the complex plane, see Figure 2.1. The contour is composed of four curves $\gamma = \gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \gamma_4$ with $x \in (\lambda, 1/\lambda)$ and

$$\begin{aligned}
\gamma_1 &: \varphi \mapsto xe^{i\varphi}, \quad \varphi \in (-\pi,\pi] \\
\gamma_2 &: \varphi \mapsto \epsilon e^{-i\varphi}, \quad \varphi \in (-\pi,\pi] \\
\gamma_3 &: t \mapsto t + i\tilde{\epsilon}, \quad t \in [-x, 0] \\
\gamma_4 &: t \mapsto -t - i\tilde{\epsilon}, \quad t \in [0, x].
\end{aligned}$$

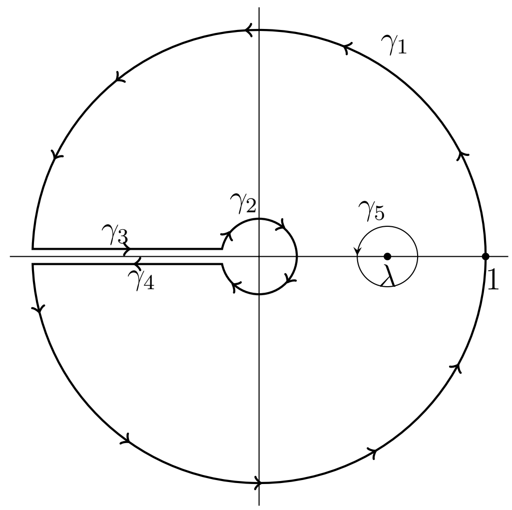


FIGURE 2.1. Contours $\gamma = \gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \gamma_4$ and γ_5 for $x = 1$

Due to holomorphy we have that

$$\frac{1}{2\pi i} \int_{\gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \gamma_4} f \, \mathrm{d}z = \frac{1}{2\pi i} \int_{\gamma_5} f \, \mathrm{d}z,$$

where γ_5 denotes a small circle around the pole λ, see Figure 2.1,

$$\gamma_5 : \varphi \mapsto \lambda + s e^{i\varphi}, \ \varphi \in (-\pi, \pi].$$

The lemma is proved by computing each of the five integrals individually.

- We first prove that

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma_1} f \, \mathrm{d}z &= \frac{1}{2\pi} \int_{-\pi}^{\pi} z^{(a+1)n+\alpha+1} \frac{(1-\lambda z)^{n+\beta}}{(z-\lambda)^{n+1}} \bigg|_{z = x e^{i\varphi}} \mathrm{d}\varphi \\ &= \frac{1}{\pi} \Re \left\{ \int_0^{\pi} z^{(a+1)n+\alpha+1} \frac{(1-\lambda z)^{n+\beta}}{(z-\lambda)^{n+1}} \bigg|_{z = x e^{i\varphi}} \mathrm{d}\varphi \right\}. \end{aligned}$$

- The standard estimate for contour integrals shows that

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_{\gamma_2} f \, \mathrm{d}z \right| &\le \varepsilon \max_{z \in \gamma_2} \left| z^{(a+1)n+\alpha+1} \frac{(1-\lambda z)^{n+\beta}}{(z-\lambda)^{n+1}} \right| \\ &\le \varepsilon \max_{z \in \gamma_2} \left| \frac{(1-\lambda z)^{\beta} z^{\alpha+1}}{z - \lambda} \right| \varepsilon^{(a+1)n} \left(\frac{1 - \varepsilon\lambda}{\lambda - \varepsilon} \right)^n, \end{aligned}$$

  where we have used [JG, Formula (1.11)] for the second inequality. It is clear that the last term goes to 0 as $\varepsilon \to 0$.

- We use that $z^y = e^{y\log(z)} = e^{y|z|+iy\arg(z)}$ with the main branch of log and evaluate

$$\int_{\gamma_3} f \; \mathrm{d}z = \int_{-x}^{0} z^{(a+1)n+\alpha}\frac{(1-\lambda z)^\beta}{z-\lambda}\left(\frac{1-\lambda z}{z-\lambda}\right)^n\Bigg|_{z=t+i\tilde{\varepsilon}} \mathrm{d}t$$

$$\xrightarrow{\tilde{\varepsilon}\to 0} \int_{-x}^{0} |t|^{(a+1)n+\alpha} e^{((a+1)n+\alpha)\pi i}\frac{|1-\lambda t|^\beta}{|t|e^{\pi i}-\lambda}\left(\frac{1-\lambda|t|e^{\pi i}}{|t|e^{\pi i}-\lambda}\right)^n \mathrm{d}t$$

$$= e^{((a+1)n+\alpha)\pi i}(-1)^{n+1}\int_{-x}^{0} |t|^{(a+1)n+\alpha}\frac{|1-\lambda t|^\beta}{|t|+\lambda}\left(\frac{1+\lambda|t|}{|t|+\lambda}\right)^n \mathrm{d}t$$

$$= e^{((a+1)n+\alpha)\pi i}(-1)^{n+1}\int_{0}^{x} t^{(a+1)n+\alpha}\frac{(1+\lambda t)^\beta}{t+\lambda}\left(\frac{1+\lambda t}{t+\lambda}\right)^n \mathrm{d}t.$$

  Similarly we find that

$$\int_{\gamma_4} f \; \mathrm{d}z = -\int_{0}^{x} z^{(a+1)n+\alpha}\frac{(1-\lambda z)^\beta}{z-\lambda}\left(\frac{1-\lambda z}{z-\lambda}\right)^n\Bigg|_{z=-t-i\tilde{\varepsilon}} \mathrm{d}t$$

$$\xrightarrow{\tilde{\varepsilon}\to 0} -\int_{0}^{x} |t|^{(a+1)n+\alpha} e^{-((a+1)n+\alpha)\pi i}\frac{|1+\lambda t|^\beta}{|t|e^{-\pi i}-\lambda}\left(\frac{1-\lambda|t|e^{-\pi i}}{|t|e^{-\pi i}-\lambda}\right)^n \mathrm{d}t$$

$$= -e^{-((a+1)n+\alpha)\pi i}(-1)^{n+1}\int_{0}^{x} t^{(a+1)n+\alpha}\frac{(1+\lambda t)^\beta}{t+\lambda}\left(\frac{1+\lambda t}{t+\lambda}\right)^n \mathrm{d}t.$$

  Summing the two terms we find that

$$\frac{1}{2\pi i}\int_{\gamma_3\oplus\gamma_4} f \; \mathrm{d}z$$

$$\xrightarrow{\tilde{\varepsilon}\to 0}(-1)^{n+1}\frac{\sin\left((a+1)n+\alpha\right)\pi}{\pi}\int_{0}^{x} t^{(a+1)n+\alpha}\frac{(1+\lambda t)^\beta}{t+\lambda}\left(\frac{1+\lambda t}{t+\lambda}\right)^n \mathrm{d}t.$$

- To evaluate the integral over γ_5 we note that f is holomorphic in a dotted neighborhood around λ. We compute

$$\frac{1}{2\pi i}\int_{\gamma_5} f \; \mathrm{d}z = \frac{s^{-n}}{2\pi}\int_{-\pi}^{\pi}(\lambda+se^{i\phi})^{(a+1)n+\alpha}(1-\lambda^2-s\lambda e^{i\varphi})^{n+\beta}e^{-in\varphi}\mathrm{d}\varphi.$$

  For $s\in(0,\,\lambda)$

$$(\lambda+se^{i\phi})^{\alpha+(a+1)n} = \sum_{\mu=0}^{\infty}\binom{\alpha+(a+1)n}{\mu}\lambda^{\alpha+(a+1)n-\mu}(se^{i\phi})^\mu,$$

  and for $s\in(0,\,\frac{1-\lambda^2}{\lambda})$

$$(1-\lambda^2-s\lambda e^{i\phi})^{n+\beta} = \sum_{\nu=0}^{\infty}\binom{n+\beta}{\nu}(1-\lambda^2)^{n+\beta-\nu}(-\lambda s e^{i\phi})^\nu,$$

and we find that

$$\frac{1}{2\pi i}\int_{\gamma_5} f\ \mathrm{d}z =$$

$$\frac{s^{-n}}{2\pi}\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\binom{\alpha+(a+1)n}{\mu}\binom{n+\beta}{\nu}\lambda^{\alpha+(a+1)n-\mu}(-\lambda)^{\nu}s^{\mu}(1-\lambda^2)^{n+\beta-\nu}s^{\nu}$$

$$\cdot\int_{-\pi}^{\pi} e^{-i(\nu+\mu-n)\phi}\frac{\mathrm{d}\phi}{2\pi}.$$

The last integral is 0 unless $\nu+\mu-n=0$ so that we set $\nu=n-\mu$ and find

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\gamma_5} f\ \mathrm{d}z \\
&=\sum_{\mu=0}^{n}\binom{\alpha+(a+1)n}{\mu}\binom{n+\beta}{n-\mu}\lambda^{\alpha+(a+1)n-\mu}(-\lambda)^{n-\mu}(1-\lambda^2)^{\beta+\mu} \\
&=\lambda^{\alpha+an}(1-\lambda^2)^{\beta}\sum_{\mu=0}^{n}\binom{\alpha+(a+1)n}{\mu}\binom{n+\beta}{n-\mu}(-\lambda^2)^{n-\mu}(1-\lambda^2)^{\mu} \\
&=\lambda^{an+\alpha}(1-\lambda^2)^{\beta}P_n^{(an+\alpha,\beta)}(1-2\lambda^2).
\end{aligned}$$

□

3. Asymptotic analysis of integral representation

3.1. **Generalized Fourier integral via the method of stationary phase.** This section is devoted to the study of the large n behavior of the integral

$$\int_0^{\pi} z^{\alpha+1}\frac{(1-\lambda z)^{\beta}}{z-\lambda}\left(z^{a+1}\frac{1-\lambda z}{z-\lambda}\right)^{n}\Bigg|_{z=e^{i\varphi}}\mathrm{d}\varphi$$

when $a\in[-\frac{2\lambda}{1+\lambda},\frac{2\lambda}{1+\lambda}]$. This is achieved using standard methods from the theory of generalized Fourier integrals. The latter are integrals of the form

$$\int_a^b g(t)e^{inh(t)}\mathrm{d}t,$$

with g and h are continuous real functions. The integral is perfectly suited for an application of the method of stationary phase. As $\left|z^{a+1}\frac{1-\lambda z}{z-\lambda}\right|=1$ for any $z=e^{i\varphi}$ we can write

$$z^{a+1}\frac{1-\lambda z}{z-\lambda}=\exp\left(ih(\varphi)\right)$$

with a real function h, which for $z=e^{i\varphi}$ is defined as

$$h(\varphi)=-i\log\left(\frac{z^{a+1}(1-\lambda z)}{z-\lambda}\right)=\operatorname{Arg}\left(\frac{z^{a+1}(1-\lambda z)}{z-\lambda}\right).$$

In this paper log always means the principal branch of the complex logarithm.

Proposition 4. *Let $\lambda \in (0,1)$, $a \in [-\frac{2\lambda}{1+\lambda}, \frac{2\lambda}{1-\lambda}]$ and $\alpha, \beta \in \mathbb{R}$. We have the following asymptotic expansion as $n \to \infty$:*

(1) *If $a \in (-\frac{2\lambda}{1+\lambda}, \frac{2\lambda}{1-\lambda})$ then*

$$\frac{1}{\pi}\Re\left\{\int_0^\pi z^{\alpha+1}\frac{(1-\lambda z)^\beta}{z-\lambda}\left(z^{a+1}\frac{1-\lambda z}{z-\lambda}\right)^n\Bigg|_{z=e^{i\varphi}}\mathrm{d}\varphi\right\}$$
$$=\sqrt{\frac{2}{n\pi}}\frac{(1-\lambda^2)^{\frac{\beta}{2}}}{(a+1)^{\frac{\beta}{2}}}\frac{\cos\left((n+1)h(\varphi_+)+(\alpha-a)\varphi_+ +(\beta-1)\psi+\frac{\pi}{4}\right)}{[(1-\lambda^2)((a+2)\lambda+a)((a+2)\lambda-a)]^{\frac{1}{4}}}$$
$$+\frac{\sin\left((an+\alpha)\pi\right)}{n\pi}\frac{(1+\lambda)^\beta}{a(1+\lambda)+2\lambda}+O(n^{-3/2})$$

where the parameters $\varphi_+, \psi \in [0,\pi]$ are defined via the relations

$$e^{i\varphi_+}=\frac{a+a\lambda^2+2\lambda^2}{2\lambda(a+1)}+i\sqrt{1-\left(\frac{a+a\lambda^2+2\lambda^2}{2\lambda(a+1)}\right)^2}$$

and

$$\sqrt{\frac{1-\lambda^2}{a+1}}e^{i\psi}=1-\lambda z_+.$$

(2) *If $a=\frac{2\lambda}{1-\lambda}$ then*

$$\frac{1}{\pi}\Re\left\{\int_0^\pi z^{\alpha+1}\frac{(1-\lambda z)^\beta}{z-\lambda}\left(z^{a+1}\frac{1-\lambda z}{z-\lambda}\right)^n\Bigg|_{z=e^{i\varphi}}\mathrm{d}\varphi\right\}$$
$$=\frac{(1-\lambda)^\beta}{3^{2/3}\Gamma(2/3)n^{1/3}\lambda^{1/3}(1+\lambda)^{1/3}}\left(1+\mathcal{O}(n^{-1/3})\right),$$

and if $a=-\frac{2\lambda}{1+\lambda}$ then

$$\frac{1}{\pi}\Re\left\{\int_0^\pi z^{\alpha+1}\frac{(1-\lambda z)^\beta}{z-\lambda}\left(z^{a+1}\frac{1-\lambda z}{z-\lambda}\right)^n\Bigg|_{z=e^{i\varphi}}\mathrm{d}\varphi\right\}$$
$$=\frac{1}{\pi}\cos\left(\left(an+\alpha+\frac{1}{6}\right)\pi\right)\frac{\Gamma(1/3)(1+\lambda)^\beta}{3^{2/3}}\left(\frac{1}{n\lambda(1-\lambda)}\right)^{1/3}\left(1+\mathcal{O}(n^{-1/3})\right).$$

Proof. We apply the method of stationary phase [AE] [BO, Chapter 6.5] to the generalized Fourier integral

$$\int_0^\pi z^{\alpha+1}\frac{(1-\lambda z)^\beta}{z-\lambda}\left(z^{a+1}\frac{1-\lambda z}{z-\lambda}\right)^n\Bigg|_{z=e^{i\varphi}}\mathrm{d}\varphi=\int_0^\pi g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi,$$

where for $z=e^{i\varphi}$ we define the functions

$$\varphi\mapsto g(\varphi)=z^{\alpha+1}\frac{(1-\lambda z)^\beta}{z-\lambda},$$
$$\varphi\mapsto h(\varphi)=-i\log\left(\frac{z^{a+1}(1-\lambda z)}{z-\lambda}\right)=\mathrm{Arg}\left(\frac{z^{a+1}(1-\lambda z)}{z-\lambda}\right).$$

Computing derivatives (in the sense of the chain rule) leads to

$$\begin{aligned}
i\frac{dh}{dz} &= -\frac{1}{z-\lambda} + \frac{a+1}{z} - \frac{\lambda}{1-\lambda z},\\
i\frac{d}{dz}\left(\frac{dh}{dz}\right) &= \frac{1}{(z-\lambda)^2} - \frac{a+1}{z^2} - \frac{\lambda^2}{(1-\lambda z)^2},\\
i\frac{d^3h}{dz^3} &= -\frac{2}{(z-\lambda)^3} + \frac{2(a+1)}{z^3} - \frac{2\lambda^3}{(1-\lambda z)^3}.
\end{aligned}$$

The function $h(\varphi)$ has a stationary point if and only if $dh/dz = 0$, i.e. iff

$$a = \frac{\lambda}{z-\lambda} + \frac{\lambda z}{1-\lambda z}.$$

Solving the latter for z gives

$$z_{+,-} = \frac{a + a\lambda^2 + 2\lambda^2}{2\lambda(a+1)} \pm i\sqrt{1 - \left(\frac{a + a\lambda^2 + 2\lambda^2}{2\lambda(a+1)}\right)^2} \in \partial\mathbb{D}$$

and we write $z_{+,-} = e^{i\varphi_{+,-}}$ with $\varphi_+ \in [0,\pi]$ and $\varphi_- \in (-\pi, 0]$. Only z_+ is interesting because we integrate over $[0,\pi]$. For the second derivative we have that

$$h''(\varphi) = \frac{d}{d\varphi}\left(\frac{dh}{dz}\frac{dz}{d\varphi}\right) = \frac{d^2h}{dz^2}\left(\frac{dz}{d\varphi}\right)^2 + \frac{dh}{dz}\frac{d^2z}{(d\varphi)^2}.$$

We distinguish the two cases *1)* $a \in (-\frac{2\lambda}{1+\lambda}, \frac{2\lambda}{1-\lambda})$ and *2)* $a \in \left\{-\frac{2\lambda}{1+\lambda}, \frac{2\lambda}{1-\lambda}\right\}$, which are characterized by the presence of a stationary point of order one ($h'(\varphi_+) = 0$ but $h''(\varphi_+) \neq 0$) in *Case 1)* and of order two ($h'(\varphi_+) = h''(\varphi_+) = 0$ but $h'''(\varphi_+) \neq 0$) in *Case 2)*.

Case 1) If $a \in (-\frac{2\lambda}{1+\lambda}, \frac{2\lambda}{1-\lambda})$ then the zeros $z_+ = e^{i\varphi_+}$ and $z_- = e^{i\varphi_-}$ of h' are distinct points located on $\partial\mathbb{D}$ with $\varphi_+ \in [0,\pi]$ and $\varphi_- \in (-\pi, 0]$. Plugging in we see that

$$i\left.\frac{d^2h}{dz^2}\right|_{z=z_+} = \frac{(1-\lambda^2)(1-z_+^2)\lambda}{z_+(z_+-\lambda)^2(1-\lambda z_+)^2}$$

so that $h''(\varphi_+) > 0$. To find the asymptotics we apply a standard result by A. Erdélyi [AE, Theorem 4] (see also F. Olver [FO1, Theorem 1] for a more explicit form), which however requires that the stationary point is an endpoint of the interval of integration. Hence we begin by splitting

$$\int_0^\pi g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi = \int_0^{\varphi_+} g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi + \int_{\varphi_+}^\pi g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi.$$

For the second integral [AE, Theorem 4] (see also [FO1, Theorem 1]) gives

$$\begin{aligned}
&\int_{\varphi_+}^{\pi} g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi \\
&= \frac{1}{2}\Gamma(1/2)k(0)e^{i\frac{\pi}{4}}n^{-1/2}e^{inh(\varphi_+)} \\
&+ \frac{1}{2}\Gamma(1)k'(0)e^{i\frac{\pi}{2}}n^{-1}e^{inh(\varphi_+)} \\
&- \frac{i}{n}e^{inh(\pi)}\frac{g(\pi)}{h'(\pi)} + \mathcal{O}\left(n^{3/2}\right),
\end{aligned}$$

with

$$k(0) = 2^{1/2}g(\varphi_+)\left(h''(\varphi_+)\right)^{-1/2}$$

and

$$k'(0) = \frac{2}{h''(\varphi_+)}g'(\varphi_+) - \frac{2}{h''(\varphi_+)}\frac{h^{(3)}(\varphi_+)}{3h''(\varphi_+)}g(\varphi_+).$$

For the first integral $\int_0^{\varphi_+} g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi$ we change the variable of integration $\varphi \mapsto -\varphi$ as suggested in [AE, page 23]. We get

$$\int_0^{\varphi_+} g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi = \int_{-\varphi_+}^{0} g(-\varphi)e^{inh(-\varphi)}\mathrm{d}\varphi.$$

Applying [AE, Theorem 4] (see also [FO1, Theorem 1]) gives

$$\begin{aligned}
&\int_{-\varphi_+}^{0} g(-\varphi)e^{inh(-\varphi)}\mathrm{d}\varphi \\
&= \frac{1}{2}\Gamma(1/2)k(0)e^{i\frac{\pi}{4}}n^{-1/2}e^{inh(\varphi_+)} \\
&+ \frac{1}{2}\Gamma(1)k'(0)e^{i\frac{\pi}{2}}n^{-1}e^{inh(\varphi_+)} \\
&- \frac{i}{n}e^{inh(0)}\frac{g(0)}{h'(0)} + \mathcal{O}\left(n^{3/2}\right)
\end{aligned}$$

with

$$k(0) = 2^{1/2}g(\varphi_+)\left(h''(\varphi_+)\right)^{-1/2}$$

and

$$k'(0) = -\frac{2}{h''(\varphi_+)}g'(\varphi_+) + \frac{2}{h''(\varphi_+)}\frac{h^{(3)}(\varphi_+)}{3h''(\varphi_+)}g(\varphi_+).$$

Observing that $h(0) = 0$ while $h(\pi) = a\pi$, $g(0) = (1-\lambda)^{\beta-1}$ while $g(\pi) = e^{i(\alpha+1)\pi}(1+\lambda)^{\beta-1}$, and $h'(0) = \frac{a(1-\lambda)-2\lambda}{1-\lambda}$ while $h'(\pi) = -\frac{a(1+\lambda)+2\lambda}{1+\lambda}$ we get

$$-\frac{i}{n}e^{inh(\pi)}\frac{g(\pi)}{h'(\pi)} = \frac{i}{n}e^{i(an+\alpha)\pi}\frac{(1+\lambda)^{\beta}}{a(1+\lambda)+2\lambda}$$

while

$$-\frac{i}{n}e^{inh(0)}\frac{g(0)}{h'(0)} = -\frac{i}{n}\frac{(1-\lambda)^{\beta}}{a(1-\lambda)-2\lambda}$$

and we conclude that

$$\begin{aligned}
&\int_0^{\pi} g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi\\
&= \Gamma(1/2)\left(2^{1/2}g(\varphi_+)\left(h''(\varphi_+)\right)^{-1/2}\right)e^{i\frac{\pi}{4}}n^{-1/2}e^{inh(\varphi_+)}\\
&+\frac{i}{n}e^{i(an+\alpha)\pi}\frac{(1+\lambda)^{\beta}}{a(1+\lambda)+2\lambda}-\frac{i}{n}\frac{(1-\lambda)^{\beta}}{a(1-\lambda)-2\lambda}+\mathcal{O}\left(n^{-3/2}\right)\\
&= e^{inh(\varphi_+)+i\frac{\pi}{4}}z_+^{\alpha+1}\frac{(1-\lambda z_+)^{\beta}}{z_+-\lambda}\left(\frac{2|z_+-\lambda|^4}{n\lambda(1-\lambda^2)|1-z_+^2|}\right)^{1/2}\Gamma(1/2)\\
&+\frac{1}{n}e^{i\left((an+\alpha)\pi+\frac{\pi}{2}\right)}\frac{(1+\lambda)^{\beta}}{a(1+\lambda)+2\lambda}-\frac{i}{n}\frac{(1-\lambda)^{\beta}}{a(1-\lambda)-2\lambda}+\mathcal{O}\left(n^{-3/2}\right).
\end{aligned}$$

We set $\psi=\arg\left(1-\lambda z_+\right)$ and get

$$\begin{aligned}
&e^{inh(\varphi_+)+i\frac{\pi}{4}}z_+^{\alpha+1}\frac{(1-\lambda z_+)^{\beta}}{z_+-\lambda}\left(\frac{2|z_+-\lambda|^4}{n\lambda(1-\lambda^2)|1-z_+^2|}\right)^{1/2}\Gamma(1/2)\\
&=\frac{\sqrt{2\pi}|1-\lambda z_+|^{\beta-1+2}}{\sqrt{n\lambda(1-\lambda^2)|1-z_+^2|}}e^{i\left(nh(\varphi_+)+\frac{\pi}{4}+(\alpha-a)\varphi_++(\beta-1)\psi+h(\varphi_+)\right)}\left(1+O(n^{-1})\right)\\
&=\sqrt{\frac{2\pi}{n}}\frac{(1-\lambda^2)^{\frac{\beta}{2}-\frac{1}{4}}}{(a+1)^{\frac{\beta}{2}}}\frac{e^{i\left((n+1)h(\varphi_+)+(\alpha-a)\varphi_++(\beta-1)\psi+\frac{\pi}{4}\right)}}{\left(((a+2)\lambda+a)((a+2)\lambda-a)\right)^{\frac{1}{4}}}\left(1+O(n^{-1})\right),
\end{aligned}$$

where we made use of

$$|1-\lambda z_+|=|z_+-\lambda|=\sqrt{\frac{1-\lambda^2}{a+1}}$$

and

$$|z_+^2-1|=2\sqrt{1-\left(\frac{a+a\lambda^2+2\lambda^2}{2\lambda(a+1)}\right)^2}=\frac{\sqrt{(1-\lambda^2)((a+2)\lambda+a)((a+2)\lambda-a)}}{\lambda(a+1)}.$$

Taking the real part of $\int_0^{\pi} g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi$ completes the proof of Theorem 4, point 1) since the real part of $\frac{1}{n}e^{i\left((an+\alpha)\pi+\frac{\pi}{2}\right)}\frac{(1+\lambda)^{\beta}}{a(1+\lambda)+2\lambda}$ is equal to

$$\frac{\sin\left((an+\alpha)\pi\right)}{n}\frac{(1+\lambda)^{\beta}}{a(1+\lambda)+2\lambda}.$$

Case 2) If $a\in\{-\frac{2\lambda}{1+\lambda},\,\frac{2\lambda}{1-\lambda}\}$ then h' has a unique zero. If $a=\frac{2\lambda}{1-\lambda}$ then $z_+=1$ and

$$h(0)=h'(0)=h''(0)=0,$$

while

$$h^{(3)}(0)=\frac{2\lambda(1+\lambda)}{(1-\lambda)^3}.$$

Applying [AE, Theorem 4] at the second order in this case yields the asymptotic behavior

$$\int_0^{\pi} g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi=g(\varphi_+)e^{inh(\varphi_+)+i\frac{\pi}{6}}\left(\frac{6}{n|h^{(3)}(\varphi_+)|}\right)^{1/3}\frac{\Gamma(1/3)}{3}\left(1+O(n^{-1/3})\right).$$

Direct computation shows that

$$\int_0^\pi g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi = z_+^{\alpha+1}\frac{(1-\lambda z_+)^\beta}{z_+-\lambda}e^{i\frac{\pi}{6}}\left(\frac{z_+^{a+1}(1-\lambda z_+)}{z_+-\lambda}\right)^n\left(\frac{3(1-\lambda)^3}{n\lambda(1+\lambda)}\right)^{1/3}\frac{\Gamma(1/3)}{3}\big(1+O(n^{-1/3})\big)$$
$$= e^{i\frac{\pi}{6}}(1-\lambda)^\beta\left(\frac{3}{n\lambda(1+\lambda)}\right)^{1/3}\frac{\Gamma(1/3)}{3}\big(1+O(n^{-1/3})\big),$$

and we conclude that

$$\frac{1}{\pi}\Re\left\{\int_0^\pi z^{\alpha+1}\frac{(1-\lambda z)^\beta}{z-\lambda}\left(z^{a+1}\frac{1-\lambda z}{z-\lambda}\right)^n\Bigg|_{z=e^{i\varphi}}\mathrm{d}\varphi\right\} = \frac{(1-\lambda)^\beta}{3^{2/3}\Gamma(2/3)}\left(\frac{1}{n\lambda(1+\lambda)}\right)^{1/3}\big(1+O(n^{-1/3})\big).$$

If $a=-\frac{2\lambda}{1+\lambda}$ then $z_+=-1$ Observe first that

$$\int_{-\pi}^0 g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi = \overline{\int_0^\pi g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi}$$

so that the saddle point $\varphi_+=-\pi$ is the left endpoint in the interval of integration; indeed we have

$$h'(-\pi)=h''(-\pi)=0, \qquad h^{(3)}(-\pi)=-\frac{2\lambda(1-\lambda)}{(1+\lambda)^3}.$$

Applying again [AE, Theorem 4] in this case yields the asymptotic behavior

$$\int_{-\pi}^0 g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi = g(-\pi)e^{inh(-\pi)-i\frac{\pi}{6}}\left(\frac{6}{n|h^{(3)}(-\pi)|}\right)^{1/3}\frac{\Gamma(1/3)}{3}\big(1+O(n^{-1/3})\big).$$

Now

$$g(-\pi)=e^{-i\pi(\alpha+1)}\frac{(1+\lambda)^\beta}{-(1+\lambda)}=(1+\lambda)^{\beta-1}e^{-i\alpha\pi}$$
$$h(-\pi)=\mathrm{Arg}\left(\frac{e^{-i\pi(a+1)}(1+\lambda)}{-(1+\lambda)}\right)=-ia\pi.$$

This gives

$$\int_{-\pi}^0 g(\varphi)e^{inh(\varphi)}\mathrm{d}\varphi = (1+\lambda)^{\beta-1}e^{-i\alpha\pi}e^{-ina\pi-i\frac{\pi}{6}}\left(\frac{3(1+\lambda)^3}{n\lambda(1-\lambda)}\right)^{1/3}\frac{\Gamma(1/3)}{3}\big(1+O(n^{-1/3})\big)$$

and we conclude that

$$\frac{1}{\pi}\Re\left\{\int_0^\pi z^{\alpha+1}\frac{(1-\lambda z)^\beta}{z-\lambda}\left(z^{a+1}\frac{1-\lambda z}{z-\lambda}\right)^n\Bigg|_{z=e^{i\varphi}}\mathrm{d}\varphi\right\}$$
$$=\frac{1}{\pi}\cos\left(\left(an+\alpha+\frac{1}{6}\right)\pi\right)\frac{\Gamma(1/3)(1+\lambda)^\beta}{3^{2/3}}\left(\frac{1}{n\lambda(1-\lambda)}\right)^{1/3}\big(1+O(n^{-1/3})\big).$$

$\square$

3.2. **The Laplace type integral via Laplace's method.** This section is devoted to the study of the large n behavior of the integral

$$\int_0^1 \frac{(1+\lambda t)^\beta t^\alpha}{t+\lambda}\left(t^{(a+1)}\frac{1+\lambda t}{t+\lambda}\right)^n \mathrm{d}t,$$

which contributes to the asymptotics of Jacobi polynomials with varying parameters iff $\alpha + an$ is not an integer. This is achieved using standard methods [BO, FO0] from the theory of Laplace-type integrals. The latter are integrals of the form

$$\int_0^1 f(t)e^{nh(t)}\mathrm{d}t,$$

where f and h are real continuous functions. In essence the idea is that if the real continuous function h has a maximum at a point $\xi \in [0,1]$ and if $f(\xi) \neq 0$ then as n grows large only values in an immediate neighborhood of ξ should contribute to the integral, see [BO, FO0]. To apply the method we introduce for $\alpha,\ \beta \in \mathbb{R}$, $a > -1$ and $\lambda \in (0,1)$ (overwriting previous notation) the real functions

$$\begin{aligned}
t &\mapsto f(t) = \frac{(1+\lambda t)^\beta t^\alpha}{t+\lambda},\ t \in [0,1] \\
t &\mapsto g(t) = t^{a+1}\frac{1+\lambda t}{\lambda+t},\ t \in [0,1] \\
t &\mapsto h(t) = \ln\left(g(t)\right),\ t \in (0,1].
\end{aligned}$$

With this notation we can write out the integral in Laplace form

$$\int_0^1 \frac{(1+\lambda t)^\beta t^\alpha}{t+\lambda}\left(t^{(a+1)}\frac{1+\lambda t}{t+\lambda}\right)^n \mathrm{d}t = \int_0^1 f(t)\left(g(t)\right)^n \mathrm{d}s = \int_0^1 f(t)e^{nh(t)}\mathrm{d}t.$$

The first step is to gain a picture about the behavior of h on $(0,1]$. Computing derivatives we find

$$\begin{aligned}
h'(t) &= \frac{a+1}{t} - \frac{1}{t+\lambda} + \frac{\lambda}{1+\lambda t}, \\
h''(t) &= -\frac{a+1}{t^2} + \frac{1}{(t+\lambda)^2} - \frac{\lambda^2}{(1+\lambda t)^2}, \\
h'''(t) &= \frac{2a+2}{t^3} - \frac{2}{(t+\lambda)^3} + \frac{2\lambda^3}{(1+\lambda t)^3}.
\end{aligned}$$

We note that $h'(t) = 0$ holds iff

$$a = \frac{t}{\lambda+t} - \frac{\lambda t}{1+\lambda t} - 1.$$

For t this yields two solutions

$$t_{+,-} = -\frac{a+a\lambda^2+2\lambda^2}{2\lambda(a+1)} \pm \sqrt{\left(\frac{a+a\lambda^2+2\lambda^2}{2\lambda(a+1)}\right)^2 - 1},$$

which are real if and only if ($a \geq \frac{2\lambda}{1-\lambda}$ *or* $a \leq \frac{-2\lambda}{1+\lambda}$).

Lemma 5. *Let $h(t) = \ln\left(t^{a+1}\frac{1+t\lambda}{t+\lambda}\right)$, $t \in (0,1]$, $\lambda \in (0,1)$ and let $a > -1$. For h we have that $\lim_{t\to 0^+} h(t) = -\infty$ and $h(1) = 0$ and the following properties*

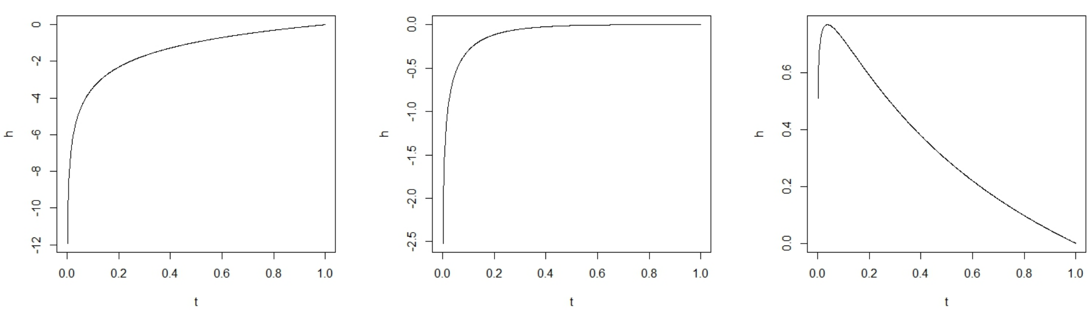
(1) *If $a > \frac{-2\lambda}{1+\lambda}$ then $h(t) \leq 0$ and $h'(t) > 0$ for $t \in (0,\, 1]$.*

(2) *If $a = \frac{-2\lambda}{1+\lambda}$ then $t_{+,-} = 1$ and $h(t) < 0$ and $h'(t) > 0$ for $t \in (0,\, 1)$. Further $h'(1) = h''(1) = 0$ and $h^{(3)}(1) = \frac{2\lambda(1-\lambda)}{(1+\lambda)^3}$.*

(3) *If $a \in (-1,\, \frac{-2\lambda}{1+\lambda})$ then h has a unique maximum at $t_- \in (0,\, 1)$ with*

$$\begin{aligned}
&h(t_-) > 0,\\
&h'(t_-) = 0,\\
&h''(t_-) = \frac{\lambda}{t_-}\left(-\frac{1}{(\lambda+t_-)^2} + \frac{1}{(1+\lambda t_-)^2}\right) < 0.
\end{aligned}$$

For illustration Figure 3.1 depicts the three cases of the lemma.



(A) (B) (C)

FIGURE 3.1. This figure depicts the situation described in Lemma 5. In all plots $\lambda = 0.3$ is chosen. A) corresponds to point (1) in the lemma and graph shows, $a = 0.9 > -6/13$. B) corresponds to point (2) in the lemma and graphs show $a = -6/13$. C) corresponds to point (3) in the lemma and graphs show $a = -0.9 < -6/13$.

Proof. The proof follows from the formulas for the derivatives of h.

(1) We have already seen that $h'(t) = 0$ holds iff

$$a = \frac{t}{\lambda+t} - \frac{\lambda t}{1+\lambda t} - 1.$$

The function $\frac{t}{\lambda+t} - \frac{\lambda t}{1+\lambda t} - 1$ is increasing on $[0,1]$ and achieves the maximum $-\frac{2\lambda}{1+\lambda}$ at $t = 1$. Hence, if $a > \frac{-2\lambda}{1+\lambda}$ then the above equation has no solution $t_* \in (0,1]$. In other words h' has no zeros on the interval $(0,1]$ i.e. h' is either positive or negative on the whole interval. Since $h(0) = -\infty$ and $h(1) = 0$ it follows that h is strictly increasing on $[0,1)$ and subsequently we also have that $h(t) \leq 0$ on $(0,1]$.

(2) Plugging $a = \frac{-2\lambda}{1+\lambda}$ into above formulas shows that $t_{+,-} = 1$ and that $h'(1) = h''(1) = 0$ and $h'''(1) = \frac{2\lambda(1-\lambda)}{(1+\lambda)^3}$. As in (1) we can conclude that $t = 1$ is the only zero of h' in $(0, 1]$ so that h' is strictly positive. Furthermore it holds that $h(t) \leq 0$ on $(0, 1]$.

(3) If $a \in (-1, \frac{-2\lambda}{1+\lambda})$ then $t_- \in (0, 1)$ and $t_+ > 1$ so that h' has a unique zero $t_- \in (0, 1)$ with

$$a = \frac{t_-}{\lambda + t_-} - \frac{\lambda t_-}{1 + \lambda t_-} - 1.$$

Plugging this into our formula for $h''(t)$ we find that

$$h''(t_-) = \frac{\lambda}{t_-}\left(-\frac{1}{(\lambda + t_-)^2} + \frac{1}{(1 + \lambda t_-)^2}\right) < 0.$$

We conclude that t_- is a local maximum of h and that h is decreasing in a neighborhood of t_-. Since $\lim_{t\to 0+} h(t) = -\infty$ and $h(1) = 0$ and there are no other critical points of h we conclude that $h(t_-)$ is a global maximum of h and therefore it is positive.

□

Proposition 6. *Let $\lambda \in (0, 1)$, $a > -1$ and let α, $\beta > -1$. We have the following asymptotic behavior as $n \to \infty$:*

(1) *If $a > \frac{-2\lambda}{1+\lambda}$ then*

$$\int_0^1 \frac{(1 + \lambda t)^\beta t^\alpha}{t + \lambda}\left(t^{(a+1)}\frac{1 + \lambda t}{t + \lambda}\right)^n \mathrm{d}t = \frac{1}{n}\frac{(1 + \lambda)^\beta}{\lambda(a + 2) + a}\left(1 + \mathcal{O}(n^{-1})\right).$$

(2) *If $a = \frac{-2\lambda}{1+\lambda}$ then*

$$\int_0^1 \frac{(1 + \lambda t)^\beta t^\alpha}{t + \lambda}\left(t^{(a+1)}\frac{1 + \lambda t}{t + \lambda}\right)^n \mathrm{d}t = \frac{1}{n^{1/3}}\frac{\Gamma(1/3)}{3^{2/3}}\frac{(1 + \lambda)^\beta}{(\lambda(1 - \lambda))^{1/3}}\left(1 + \mathcal{O}(n^{-1/3})\right).$$

(3) *If $a \in (-1, \frac{-2\lambda}{1+\lambda})$ then $g(t_-) > 1$ and*

$$\int_0^1 \frac{(1 + \lambda t)^\beta t^\alpha}{t + \lambda}\left(t^{(a+1)}\frac{1 + \lambda t}{t + \lambda}\right)^n \mathrm{d}t$$
$$\sim (1 + \lambda t_-)^{\beta+1} t_-^\alpha \left(g(t_-)\right)^n \sqrt{\frac{2\pi t_-}{n\lambda\left((1 + \lambda t_-)^2 - (\lambda + t_-)^2\right)}}.$$

Proof. For properties of h we refer to the respective points of Lemma 5. We employ standard Laplace methods to study the asymptotic behavior

$$\begin{aligned}
&\int_0^1 \frac{(1 + \lambda t)^\beta (-t)^\alpha}{t + \lambda}\left(t^{(a+1)}\frac{1 + \lambda t}{t + \lambda}\right)^n \mathrm{d}t \\
&= \int_{-1}^0 \frac{(1 - \lambda t)^\beta t^\alpha}{\lambda - t}\left((-t)^{(a+1)}\frac{1 - \lambda t}{t - \lambda}\right)^n \mathrm{d}t \\
&= \int_{-1}^0 \tilde{f}(t)e^{-n\tilde{h}(t)}\mathrm{d}t,
\end{aligned}$$

where

$$\tilde{f}(t) = f(-t) = \frac{(1-\lambda t)^{\beta}(-t)^{\alpha}}{\lambda - t},$$
$$\tilde{h}(t) = -h(-t) = -\ln\left((-t)^{a+1}\frac{1-\lambda t}{\lambda - t}\right).$$

(1) If $a > \frac{-2\lambda}{1+\lambda}$ then $\tilde{h}(-1) = 0$, $\lim_{t\to 0^+}\tilde{h}(t) = +\infty$ and for $t \in [-1, 0)$ that $\tilde{h}(t) \geq 0$ and $\tilde{h}'(t) > 0$. Applying a result by A. Erdélyi (1956) [FO0, Theorem 7.1 page 81] we get

$$\begin{aligned}
&\int_{-1}^{0} \tilde{f}(t) e^{-n\tilde{h}(t)} \mathrm{d}t \\
&= \frac{\tilde{f}(-1)}{n\tilde{h}'(-1)} e^{n\tilde{h}(-1)} \left(1 + \mathcal{O}(n^{-1})\right) \\
&= \frac{1}{n}\frac{(1+\lambda)^{\beta}}{\lambda(a+2)+a} \left(1 + \mathcal{O}(n^{-1})\right)
\end{aligned}$$

which completes the proof.

(2) If $a = -\frac{2\lambda}{1+\lambda}$ then $\tilde{h}$ is positive and strictly increasing on $[-1, 0)$ with $\tilde{h}(-1) = 0$, $\lim_{t\to 0^-}\tilde{h}(t) = \infty$. Moreover $\tilde{h}'(-1) = \tilde{h}''(-1) = 0$ while $\tilde{h}^{(3)}(-1) = h^{(3)}(1) = \frac{2\lambda(1-\lambda)}{(1+\lambda)^3} > 0$. Applying again A. Erdélyi's result [FO0, Theorem 7.1 page 81] we get

$$\begin{aligned}
\int_{-1}^{0} \tilde{f}(t) e^{-n\tilde{h}(t)} \mathrm{d}t &= \tilde{f}(-1) e^{-n\tilde{h}(-1)} \frac{\Gamma(1/3)}{3} \left(\frac{3!}{n\tilde{h}^{(3)}(-1)}\right)^{1/3} \left(1 + \mathcal{O}(n^{-1/3})\right) \\
&= \frac{(1+\lambda)^{\beta-1}}{n^{1/3}} \frac{\Gamma(1/3)}{3^{2/3}} \frac{1+\lambda}{\left(\lambda(1-\lambda)\right)^{1/3}} \left(1 + \mathcal{O}(n^{-1/3})\right) \\
&= \frac{1}{n^{1/3}} \frac{\Gamma(1/3)}{3^{2/3}} \frac{(1+\lambda)^{\beta}}{\left(\lambda(1-\lambda)\right)^{1/3}} \left(1 + \mathcal{O}(n^{-1/3})\right).
\end{aligned}$$

(3) If $a \in (-1, \frac{-2\lambda}{1+\lambda})$ then t_- is the unique maximum of h on $(0, 1)$ with $g(t_-) = e^{h(t_-)} > 1$. Applying standard results about Laplace integrals [BO, formula (6.4.19c) page 267] we obtain

$$\int_0^1 \frac{(1+\lambda t)^{\beta} t^{\alpha}}{t+\lambda} \left(t^{(a+1)} \frac{1+\lambda t}{t+\lambda}\right)^n \mathrm{d}t \sim \frac{(1+\lambda t_-)^{\beta} t_-^{\alpha}}{t_- + \lambda} \sqrt{\frac{2\pi}{-nh''(t_-)}} \left(g(t_-)\right)^n,$$

as $n \to +\infty$. Moreover

$$\sqrt{\frac{2\pi}{-nh''(t_-)}} = (\lambda + t_-)(1 + \lambda t_-) \sqrt{\frac{2\pi t_-}{n\lambda\left((1+\lambda t_-)^2 - (\lambda + t_-)^2\right)}}$$

and the result follows.

□

4. Simple exponential upper estimates

In Lemma 3 we chose to not specify the parameter $x \in (\lambda, 1/\lambda)$. This allows us to tune the parameter to find simple exponential bounds for Jacobi polynomials for $a \in (-1, -\frac{2\lambda}{1+\lambda}) \cup (\frac{2\lambda}{1-\lambda}, \infty)$.

Proposition 7. *Let $\alpha,\ \beta > -1$, $a > -1$, $\lambda \in (0,\,1)$ and n be a positive integer.*

(1) *We assume that $\alpha + an$ is an integer.*

(a) *If $a > \frac{2\lambda}{1-\lambda}$ then there is $x^* \in (\lambda, 1)$ such that*

$$|\lambda^{an+\alpha}(1-\lambda^2)^\beta P_n^{(an+\alpha,\,\beta)}(1-2\lambda^2)| \le (x^*)^{\alpha-a}\,\psi_\beta(\lambda x^*)\left((x^*)^{a+1}\,\frac{1-\lambda x^*}{x^*-\lambda}\right)^{n+1}$$

and $(x^*)^{a+1}\,\frac{1-\lambda x^*}{x^*-\lambda} < 1$ where $\psi_\beta(u) = \begin{cases}(1+u)^{\beta-1} & if\ \beta \ge 1\\ (1-u)^{\beta-1}, & if\ \beta < 1\end{cases}$.
x^* is computed explicitly below in (4.2).

(b) *If $a < -\frac{2\lambda}{1+\lambda}$ then for any $x \in (1, 1/\lambda)$*

$$|\lambda^{an+\alpha}(1-\lambda^2)^\beta P_n^{(an+\alpha,\,\beta)}(1-2\lambda^2)| \le x^{\alpha-a}\psi_\beta(\lambda x)\left(x^{a+1}\frac{1-\lambda x}{x-\lambda}\right)^{n+1}$$

and $x^{a+1}\frac{1-\lambda x}{x-\lambda} < 1$.

(2) *We assume that $\alpha + an$ is not an integer.*

(a) *If $a > \frac{2\lambda}{1-\lambda}$ then there is $x^* \in (0, 1)$ such that*

$$\begin{aligned}&|\lambda^{an+\alpha}(1-\lambda^2)^\beta P_n^{(an+\alpha,\,\beta)}(1-2\lambda^2)|\\ &\le (x^*)^{\alpha-a}\,\psi_\beta(\lambda x^*)\left((x^*)^{a+1}\,\frac{1-\lambda x^*}{x^*-\lambda}\right)^{n+1}\\ &+\left(\int_0^1 \frac{(1+\lambda t)^\beta t^\alpha}{t+\lambda}\mathrm{d}t\right)\left((x^*)^{a+1}\,\frac{1+\lambda x^*}{x^*+\lambda}\right)^{n}.\end{aligned}$$

and $(x^*)^{a+1}\,\frac{1-\lambda x^*}{x^*-\lambda} < 1$, $(x^*)^{a+1}\,\frac{1+\lambda x^*}{x^*+\lambda} < 1$.

(b) *If $a < -\frac{2\lambda}{1+\lambda}$*

$$\begin{aligned}&\lambda^{an+\alpha}(1-\lambda^2)^\beta P_n^{(an+\alpha,\,\beta)}(1-2\lambda^2)\\ &\sim (1+\lambda t_-)^{\beta+1} t_-^\alpha\,(g(t_-))^n\sqrt{\frac{2\pi t_-}{n\lambda\left((1+\lambda t_-)^2-(\lambda+t_-)^2\right)}}\end{aligned}$$

where $t_- \in (0,1)$ is computed explicitly below in (3.2) and $g(t_-) = t_-^{a+1}\frac{1+\lambda t_-}{\lambda+t_-} > 1$.

To start with we find upper estimates for

$$\left|\frac{1}{\pi}\Re\left\{\int_0^\pi z^{\alpha+1}\frac{(1-\lambda z)^\beta}{z-\lambda}\left(z^{a+1}\frac{1-\lambda z}{z-\lambda}\right)^n\bigg|_{z=xe^{i\varphi}}\mathrm{d}\varphi\right\}\right|.$$

Lemma 8. *We have the following exponential upper bounds.*

(1) *If* $a > \frac{2\lambda}{1-\lambda}$ *then there is* $x^* \in (\lambda, 1)$ *such that* $(x^*)^{a+1}\,\frac{1-\lambda x^*}{x^*-\lambda} \in (0,\,1)$ *and*

$$\begin{aligned}&\left|\frac{1}{\pi}\Re\left\{\int_0^\pi z^{\alpha+1}\frac{(1-\lambda z)^\beta}{z-\lambda}\left(z^{a+1}\frac{1-\lambda z}{z-\lambda}\right)^n\right|_{z=x^*e^{i\varphi}} \mathrm{d}\varphi\right\}\Bigg|\\ &\le (x^*)^{\alpha-a}\,\psi_\beta(\lambda x^*)\left((x^*)^{a+1}\,\frac{1-\lambda x^*}{x^*-\lambda}\right)^{n+1}\end{aligned}$$

where $\psi_\beta(u) = \begin{cases}(1+u)^{\beta-1} & if\ \beta \ge 1\\ (1-u)^{\beta-1}, & if\ \beta < 1\end{cases}$. x^* *is computed explicitly below in* (4.2).

(2) *If* $-1 < a < \frac{-2\lambda}{1+\lambda}$ *then for any* $x \in (1, 1/\lambda)$ *we have* $x^{a+1}\frac{1-\lambda x}{x-\lambda} \in (0,\,1)$ *and*

$$\begin{aligned}&\left|\frac{1}{\pi}\Re\left\{\int_0^\pi z^{\alpha+1}\frac{(1-\lambda z)^\beta}{z-\lambda}\left(z^{a+1}\frac{1-\lambda z}{z-\lambda}\right)^n\right|_{z=xe^{i\varphi}} \mathrm{d}\varphi\right\}\Bigg|\\ &\le x^{\alpha-a}\psi_\beta(\lambda x)\left(x^{a+1}\frac{1-\lambda x}{x-\lambda}\right)^{n+1}.\end{aligned}$$

Proof. Clearly

$$\begin{aligned}&\left|\frac{1}{\pi}\Re\left\{\int_0^\pi z^{\alpha+1}\frac{(1-\lambda z)^\beta}{z-\lambda}\left(z^{a+1}\frac{1-\lambda z}{z-\lambda}\right)^n\right|_{z=xe^{i\varphi}} \mathrm{d}\varphi\right\}\Bigg|\\ &\le \max_{|z|=x}\left|z^{\alpha+1}\frac{(1-\lambda z)^\beta}{z-\lambda}\right|\left(\max_{|z|=x}|z^{a+1}\frac{1-\lambda z}{z-\lambda}|\right)^n.\end{aligned}$$

For $z = xe^{i\varphi}$ we have

$$\max_{|z|=x}\left|z^{a+1}\frac{1-\lambda z}{z-\lambda}\right| = x^{a+1}\max_{-\pi\le\varphi\le\pi}\left|\frac{1-\lambda xe^{i\varphi}}{xe^{i\varphi}-\lambda}\right|.$$

Direct computation shows that

$$\frac{\partial}{\partial\varphi}|\frac{1-\lambda xe^{i\varphi}}{xe^{i\varphi}-\lambda}|^2 = -\frac{2(1-x^2)(1-\lambda^2)\lambda x\sin\varphi}{|xe^{i\varphi}-\lambda|^2},$$

so the corresponding max is attained either at $\varphi = 0$ or at $\varphi = \pm\pi$. Thus $\max_{-\pi\le\varphi\le\pi}|\frac{1-\lambda xe^{i\varphi}}{xe^{i\varphi}-\lambda}|$ is either equal to $|\frac{1-\lambda x}{x-\lambda}|$ or to $\frac{1+\lambda x}{x+\lambda}$. Since $x \in (\lambda,\,1/\lambda)$ we have $|\frac{1-\lambda x}{x-\lambda}| = \frac{1-\lambda x}{x-\lambda}$ and

$$\frac{1-\lambda x}{x-\lambda} - \frac{1+\lambda x}{x+\lambda} = \frac{2\lambda(1-x^2)}{x^2-\lambda^2} > 0.$$

This gives $\max_{-\pi\le\varphi\le\pi}|\frac{1-\lambda xe^{i\varphi}}{xe^{i\varphi}-\lambda}| = \frac{1-\lambda x}{x-\lambda}$ and thus

$$\max_{|z|=x}|z^{a+1}\frac{1-\lambda z}{z-\lambda}| = x^{a+1}\frac{1-\lambda x}{x-\lambda},$$

but also

$$\max_{|z|=x} |z^{\alpha+1}\frac{(1-\lambda z)^{\beta}}{z-\lambda}| = x^{\alpha+1}\max_{|z|=x}|\frac{(1-\lambda z)^{\beta}}{z-\lambda}|$$
$$\leq x^{\alpha+1}\psi_{\beta}(\lambda x)\max_{|z|=x}|\frac{1-\lambda z}{z-\lambda}|$$
$$= x^{\alpha+1}\psi_{\beta}(\lambda x)\frac{1-\lambda x}{x-\lambda},$$

and as a consequence

$$\left|\frac{1}{\pi}\Re\left\{\int_0^{\pi} z^{\alpha+1}\frac{(1-\lambda z)^{\beta}}{z-\lambda}\left(z^{a+1}\frac{1-\lambda z}{z-\lambda}\right)^n\Big|_{z=xe^{i\varphi}} \mathrm{d}\varphi\right\}\right|$$
$$\leq x^{\alpha+1}\psi_{\beta}(\lambda x)\frac{1-\lambda x}{x-\lambda}\left(x^{a+1}\frac{1-\lambda x}{x-\lambda}\right)^n$$
$$\leq x^{\alpha-a}\psi_{\beta}(\lambda x)\left(x^{a+1}\frac{1-\lambda x}{x-\lambda}\right)^{n+1}.$$

We put $g(x) = x^{a+1}\frac{1-\lambda x}{x-\lambda}$. The derivative of g with respect to x is zero if and only if the variables λ, a, x satisfy

$$a = \frac{\lambda}{x-\lambda} + \frac{\lambda x}{1-\lambda x}.$$

Solving the latter for x gives:

$$x_{+,-} = \frac{a+a\lambda^2+2\lambda^2}{2\lambda(a+1)} \pm \sqrt{\left(\frac{a+a\lambda^2+2\lambda^2}{2\lambda(a+1)}\right)^2 - 1},$$

which is real if and only if ($a \geq \frac{2\lambda}{1-\lambda}$ *or* $a \leq \frac{-2\lambda}{1+\lambda}$). Moreover direct computation shows that

$$g''(x_{+,-}) = x_{+,-}^{a+1}\frac{\lambda(1-x_{+,-}^2)(1-\lambda^2)}{x_{+,-}(x_{+,-}-\lambda)^3(1-\lambda x_{+,-})} \tag{4.1}$$

and the fact that $x_{+,-}$ is a max or min will depend on its position with respect to 1, λ and $1/\lambda$.

1) We first prove that *if* $a > \frac{2\lambda}{1-\lambda}$ *then* $x_+ \in (1,\, 1/\lambda)$ *while* $x_- \in (\lambda,\, 1)$ *and*

$$x^* = x_- = \frac{a+a\lambda^2+2\lambda^2}{2\lambda(a+1)} - \sqrt{\left(\frac{a+a\lambda^2+2\lambda^2}{2\lambda(a+1)}\right)^2 - 1} \tag{4.2}$$

satisfies

$$g(x^*) < 1.$$

We note that $x_{+,-}(a = \frac{2\lambda}{1-\lambda}) = 1$, $x_{+,-}(a = \frac{-2\lambda}{1+\lambda}) = -1$ while $x_+(a \to \infty) \to \frac{1}{\lambda}$ and $x_-(a \to \infty) \to \lambda$. The function $a \mapsto y(a) = \frac{a+a\lambda^2+2\lambda^2}{2\lambda(a+1)}$ is monotonically increasing on $\mathbb{R}$ for $\lambda \in (0,1)$ and the function $f_+(y) = y + \sqrt{y^2-1}$ is increasing for $y > 1$. This shows that $x_+(a)$ is increasing for $a > \frac{2\lambda}{1-\lambda}$ so that $x_+ \in (1, 1/\lambda)$. Similarly $x_- \in (\lambda, 1)$ and $x_-(a)$ is decreasing for $a > \frac{2\lambda}{1-\lambda}$ since the function $f_-(y) = y - \sqrt{y^2-1}$ is decreasing for $y > 1$. For illustration Figure 4.1 depicts graphs of $f_+(y) = y + \sqrt{y^2-1}$ and $f_-(y) = y - \sqrt{y^2-1}$.

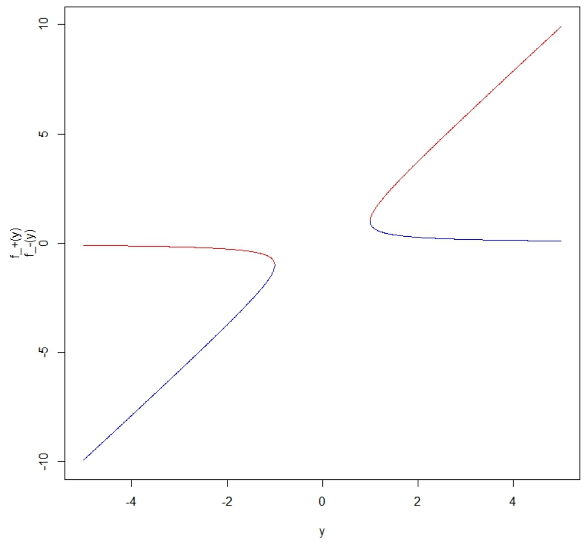


FIGURE 4.1. Plot of functions $f_+(y) = y + \sqrt{y^2-1}$ (red) and $f_-(y) = y - \sqrt{y^2-1}$ (blue).

In particular $x \mapsto g'(x)$ vanishes on $(\lambda, 1]$ if and only if $x = x_-$. Moreover the fact that $x_- \in (\lambda, 1)$ taken together with formula (4.1) gives $g''(x_-) < 0$ and x_- is a min for g on $(\lambda, 1)$. Taking into account $\lim_{x\to\lambda^+} g(x) = +\infty$ and $g(1) = 1$, g is strictly decreasing on $(\lambda, x_-]$, strictly increasing on $[x_-, 1]$ and admits a unique minimum $x^* = x_-$ on $(\lambda, 1]$ which satisfies $g(x^*) < 1$.

2) We prove that *if* $a < -\frac{2\lambda}{1-\lambda}$ *then* $x_{+,-} < 0$ *and for any* $x \in (1, 1/\lambda)$ *we have*

$$g(x) < 1.$$

Indeed we observe first that

$$-1 < a < -\frac{2\lambda}{1-\lambda} < -\frac{2\lambda^2}{1+\lambda^2}$$

because $\frac{2\lambda}{1+\lambda} - \frac{2\lambda^2}{1+\lambda^2} = \frac{2\lambda(1-\lambda)}{(1+\lambda)(1+\lambda^2)} > 0$. In particular

$$\frac{a + a\lambda^2 + 2\lambda^2}{2\lambda(a+1)} < 0$$

and thus $x_- < 0$. Moreover

$$\sqrt{\left(\frac{a + a\lambda^2 + 2\lambda^2}{2\lambda(a+1)}\right)^2 - 1} < \left|\frac{a + a\lambda^2 + 2\lambda^2}{2\lambda(a+1)}\right|$$

so that

$$\frac{a+a\lambda^2+2\lambda^2}{2\lambda(a+1)}+\sqrt{\left(\frac{a+a\lambda^2+2\lambda^2}{2\lambda(a+1)}\right)^2-1}<0.$$

In particular g' does not vanish on $(\lambda,\,1/\lambda]$ with $\lim_{x\to\lambda^+} g(x) = +\infty$, $g(1) = 1$ and $g(1/\lambda) = 0$: as a consequence g is strictly decreasing on $(1,\,1/\lambda)$ and satisfies

$$0 < g(x) < 1$$

for any $x \in (1, 1/\lambda)$. □

We conclude by proving Proposition 7.

Proof of Proposition 7. Since (1) is clear from Lemma 8 we prove (2).

(2) If $a > \frac{2\lambda}{1-\lambda}$ we apply Lemma 8 point (1) and choose $x = x^* \in (\lambda, 1)$ such that $(x^*)^{a+1}\,\frac{1-\lambda x^*}{x^*-\lambda} \in (0,\,1)$. We have

$$\begin{aligned}&\left|\frac{1}{\pi}\Re\left\{\int_0^\pi z^{\alpha+1}\frac{(1-\lambda z)^\beta}{z-\lambda}\left(z^{a+1}\frac{1-\lambda z}{z-\lambda}\right)^n\Big|_{z=x^*e^{i\varphi}}\mathrm{d}\varphi\right\}\right|\\&\le (x^*)^{\alpha-a}\,\psi_\beta(\lambda x^*)\left((x^*)^{a+1}\,\frac{1-\lambda x^*}{x^*-\lambda}\right)^{n+1}\end{aligned}$$

where $\psi_\beta(u) = \begin{cases}(1+u)^{\beta-1} & if\ \beta\ge 1\\ (1-u)^{\beta-1}, & if\ \beta<1\end{cases}$. Moreover according to Lemma 5 point (1) $g : t \mapsto t^{(a+1)}\frac{1+\lambda t}{t+\lambda}$ is strictly increasing on $[0,\,1]$ so

$$\begin{aligned}&\int_0^{x^*}\frac{(1+\lambda t)^\beta t^\alpha}{t+\lambda}\left(t^{(a+1)}\frac{1+\lambda t}{t+\lambda}\right)^n\mathrm{d}t\\&\le\left((x^*)^{a+1}\,\frac{1+\lambda x^*}{x^*+\lambda}\right)^n\int_0^{x^*}\frac{(1+\lambda t)^\beta t^\alpha}{t+\lambda}\mathrm{d}t\end{aligned}$$

with $(x^*)^{a+1}\,\frac{1+\lambda x^*}{x^*+\lambda} < 1$. Thus

$$\begin{aligned}&|\lambda^{an+\alpha}(1-\lambda^2)^\beta P_n^{(an+\alpha,\,\beta)}(1-2\lambda^2)|\\&\le (x^*)^{\alpha-a}\,\psi_\beta(\lambda x^*)\left((x^*)^{a+1}\,\frac{1-\lambda x^*}{x^*-\lambda}\right)^{n+1}+\left(\int_0^1\frac{(1+\lambda t)^\beta t^\alpha}{t+\lambda}\mathrm{d}t\right)\left((x^*)^{a+1}\,\frac{1+\lambda x^*}{x^*+\lambda}\right)^n.\end{aligned}$$

If $a < -\frac{2\lambda}{1+\lambda}$ we apply Lemma 8 point (2). For any $x \in (1, 1/\lambda)$ we have $x^{a+1}\frac{1-\lambda x}{x-\lambda} \in (0,\,1)$ and

$$\begin{aligned}&|\frac{1}{\pi}\Re\left\{\int_0^\pi z^{\alpha+1}\frac{(1-\lambda z)^\beta}{z-\lambda}\left(z^{a+1}\frac{1-\lambda z}{z-\lambda}\right)^n\Big|_{z=xe^{i\varphi}}\mathrm{d}\varphi\right\}|\\&\le x^{\alpha-a}\psi_\beta(\lambda x)\left(x^{a+1}\frac{1-\lambda x}{x-\lambda}\right)^{n+1}.\end{aligned}$$

We choose any x such that $x \in (1, \min(1/\lambda, t_+))$ where

$$t_+ = -\frac{a + a\lambda^2 + 2\lambda^2}{2\lambda(a+1)} + \sqrt{\left(\frac{a + a\lambda^2 + 2\lambda^2}{2\lambda(a+1)}\right)^2 - 1}$$

and observe that the second critical point t_+ for g (the first one being $t_- \in (\lambda, 1)$ see Lemma 5 point (3)) satisfies $t_+ > 1$. Indeed

$$t_+ - 1 = \sqrt{\left(\frac{a + a\lambda^2 + 2\lambda^2}{2\lambda(a+1)}\right)^2 - 1} - \frac{(1+\lambda)(a(1+\lambda) + 2\lambda)}{2\lambda(a+1)} > 0$$

since $a(1+\lambda) + 2\lambda < 0$. Then we write

$$\begin{aligned} &\int_0^x \frac{(1+\lambda t)^\beta t^\alpha}{t+\lambda} \left(t^{(a+1)} \frac{1+\lambda t}{t+\lambda}\right)^n \mathrm{d}t \\ &= \int_0^1 \frac{(1+\lambda t)^\beta t^\alpha}{t+\lambda} \left(t^{(a+1)} \frac{1+\lambda t}{t+\lambda}\right)^n \mathrm{d}t + \int_1^x \frac{(1+\lambda t)^\beta t^\alpha}{t+\lambda} \left(t^{(a+1)} \frac{1+\lambda t}{t+\lambda}\right)^n \mathrm{d}t \end{aligned}$$

and the asymptotic (exponential) behavior of the first integral is given in Lemma 5 point (2) while the second one is $\mathcal{O}(n^{-1})$ because x has been chosen so that g' does not vanish on $[1, x]$. Thus

$$\begin{aligned} &\lambda^{an+\alpha}(1-\lambda^2)^\beta P_n^{(an+\alpha,\,\beta)}(1-2\lambda^2) \\ &\sim \int_0^1 \frac{(1+\lambda t)^\beta t^\alpha}{t+\lambda} \left(t^{(a+1)} \frac{1+\lambda t}{t+\lambda}\right)^n \mathrm{d}t \\ &\sim (1+\lambda t_-)^{\beta+1} t_-^\alpha \left(g(t_-)\right)^n \sqrt{\frac{2\pi t_-}{n\lambda\left((1+\lambda t_-)^2 - (\lambda + t_-)^2\right)}}, \end{aligned}$$

where

$$t_- = -\frac{a + a\lambda^2 + 2\lambda^2}{2\lambda(a+1)} - \sqrt{\left(\frac{a + a\lambda^2 + 2\lambda^2}{2\lambda(a+1)}\right)^2 - 1}$$

and $g(t_-) > 1$.

□

5. Proof of Theorem 2

We add this section for completeness. We collect our results on the two integrals and write up a proof of the main theorem. Recall that according to Lemma 3 we have

$$\begin{aligned} &\lambda^{an+\alpha}(1-\lambda^2)^\beta P_n^{(an+\alpha,\,\beta)}(1-2\lambda^2) \\ &= \frac{1}{\pi}\Re\left\{\int_0^\pi z^{\alpha+1}\frac{(1-\lambda z)^\beta}{z-\lambda}\left(z^{a+1}\frac{1-\lambda z}{z-\lambda}\right)^n\Big|_{z=xe^{i\varphi}} \mathrm{d}\varphi\right\} \\ &- \frac{\sin\left(\pi(\alpha+an)\right)}{\pi}\int_0^x \frac{(1+\lambda t)^\beta t^\alpha}{t+\lambda}\left(t^{(a+1)}\frac{1+\lambda t}{t+\lambda}\right)^n \mathrm{d}t. \end{aligned}$$

Proof of Theorem 2. (1) If $a \in (-\frac{2\lambda}{1+\lambda}, \frac{2\lambda}{1-\lambda})$ we choose $x = 1$ and conclude by combining Proposition 6 point (1) and Proposition 4 point (1).

(2) If $a = \frac{2\lambda}{1-\lambda}$ we choose again $x = 1$ and conclude by combining Proposition 6 point (2) and Proposition 4 point (2). If $a = -\frac{2\lambda}{1+\lambda}$ we choose $x = 1$. According to Proposition 6 point (2) we have

$$\begin{aligned}&\frac{\sin(\pi(\alpha + an))}{\pi}\int_0^1 \frac{(1+\lambda t)^\beta t^\alpha}{t+\lambda}\left(t^{(a+1)}\frac{1+\lambda t}{t+\lambda}\right)^n \mathrm{d}t\\ &= \frac{C(\lambda,\beta)}{n^{1/3}}\sin((an+\alpha)\pi) + \mathcal{O}\left(n^{-2/3}\right)\end{aligned}$$

where $C(\lambda,\beta) = \frac{\Gamma(1/3)}{3^{2/3}\pi}\frac{(1+\lambda)^\beta}{(\lambda(1-\lambda))^{1/3}}$. Proposition 4 point (2) gives

$$\begin{aligned}&\frac{1}{\pi}\Re\left\{\int_0^\pi z^{\alpha+1}\frac{(1-\lambda z)^\beta}{z-\lambda}\left(z^{a+1}\frac{1-\lambda z}{z-\lambda}\right)^n\bigg|_{z=e^{i\varphi}}\mathrm{d}\varphi\right\}\\ &= \frac{C(\lambda,\beta)}{n^{1/3}}\cos\left(\left(an+\alpha+\frac{1}{6}\right)\pi\right) + \mathcal{O}\left(n^{-2/3}\right).\end{aligned}$$

The result follows by subtracting the two above equalities. (3) and (4) are direct consequences of Proposition 7. □

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